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Groups with Completely Regular Primitive Dual Space

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We show that if G is a unimodular, amenable group whose primitive dual space $\Pr C^*(G)$ satisfies certain regularity conditions, then G is in the class $[FC]^-$. This is a partial converse to a result of Liukkonen and Mosak that a σ-compact $[FC]^-$ group has a Hausdorff primitive dual space.

The question of what significance, if any, a Hausdorff topology on the dual space $\hat{G}$ of a locally compact group G has, was first raised by Liukkonen [7], who showed that a type I $[FC]^-$ group has a Hausdorff dual space. Removing the restrictive condition of type I-ness necessarily imposed if $\hat{G}$ is to be Hausdorff by considering primitive ideals instead of representations, Liukkonen and Mosak [8] recently showed that any σ-compact $[FC]^-$ group has a Hausdorff primitive ideal space. Our original intent was to investigate the converse proposition. As it turned out we were able to weaken the assumption that $\Pr C^*(G)$ be Hausdorff, while on the other hand we were forced to introduce another condition.

DEFINITION. Let A be a C^*-algebra and let $\Pr A$ denote the primitive ideals of A topologized by the kernel–hull topology. We will say $\Pr A$ is regular at a point $P \in \Pr A$ if there is a neighborhood U of P such that U is a Hausdorff space in the relative topology and $A/k(U)$ contains an identity. ($k(U) = \bigcap_{P' \in U} P'$.) $\Pr A$ is completely regular if it is regular at every point.

THEOREM 1. *Let G be unimodular and amenable and suppose $\Pr C^*(G)$ is regular at P_e, the kernel in $C^*(G)$ of the trivial representation of G. Then $G \in [FC]^-$.*

THEOREM 2. *Let G be unimodular and suppose $\Pr C^*(G)$ is completely regular. Then $G \in [SIN]$.*

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COROLLARY 3. *Let G be unimodular and amenable and suppose* $\mathrm{Pr}\, C^*(G)$ *is completely regular. Then* $G \in [FIA]^-$.

Proof. From Theorems 1 and 2 we have $G \in [FC]^- \cap [SIN]$. But this is precisely the class $[FIA]^-$ (see [5]).

We remark that our definition of complete regularity for $\mathrm{Pr}\, A$ is not to be confused with the usual notion of complete regularity defined for topological spaces. Technically, we should probably define regularity at a point and complete regularity for the pair $(A, \mathrm{Pr}\, A)$; for the sake of brevity, however, we will not do this. Thus when we write that $\mathrm{Pr}\, A$ is completely regular we will mean in accordance with the above definition; otherwise, if we are referring to complete regularity in the topological sense, we will write "topologically completely regular."

As above, let A be a C^*-algebra. We will show that $\mathrm{Pr}\, A$ completely regular implies $\mathrm{Pr}\, A$ topologically completely regular. Since the primitive ideal space of a C^*-algebra is always locally compact, it will be enough to show that $\mathrm{Pr}\, A$ is Hausdorff. Let $P \neq Q \in \mathrm{Pr}\, A$, and let U, V be neighborhoods of P, Q which are Hausdorff in the relative topology. Since $\mathrm{Pr}\, A$ is locally compact,[1] we may assume U and V are compact. If both P and Q are contained in either U or V we are done. Otherwise, observe that $U \cap V$ is closed, since $U \cap V$ is a compact subset of U and the latter is a Hausdorff space. Then $U \sim U \cap V$ and $V \sim U \cap V$ are disjoint neighborhoods of P and Q.

Thus if $\mathrm{Pr}\, A$ is completely regular, the algebra A is completely regular in the sense of [11]. Conversely, if A is completely regular in the sense of [11] and if each $P \in \mathrm{Pr}\, A$ is modular (so that the structure space and the strong structure space coincide), then $\mathrm{Pr}\, A$ is completely regular.

NOTATIONS. If G is a locally compact group we write $U(G)$ for the von Neumann algebra in $\mathscr{L}(L^2(G))$ generated by $\{\lambda(f) : f \in L^1(G)\}$, where λ is the left regular representation. $P_1(G)$ is the set of continuous positive definite functions of G of norm $\leqslant 1$. If G is unimodular then $L^1 \cap L^2(G)$ is a Hilbert algebra, and its fulfillment (see [12]), the associated full Hilbert algebra, will be written $A(G)$. If $H \subseteq G$ is a subset and f a function defined on H, then $\tilde{f}$ will be the function such that $\tilde{f} \mid H = f$ and $\mathrm{support}(\tilde{f}) \subseteq H$. The left and right translates ${}_xf$ and f_x are given by ${}_xf(y) = f(xy)$ and $f_x(y) = f(yx)$.

Remark 4. Let $G \in [FIA]^-$ (resp. $[FC]^-$); then $\mathrm{Pr}\, C^*(G)$ is

[1] Its topology has a basis consisting of compact sets.

completely regular (resp., regular at P_e). Thus the converses of Theorem 1 and Corollary 3 hold.

Proof. If $G \in [FIA]^-$, then $\Pr C^*(G)$ is homeomorphic with the character space $X(G)$ which is a locally compact Hausdorff space [9, 4.6 and 5.2]. Then it follows from [6, Lemma 2] that $\Pr C^*(G)$ is completely regular. Now if $G \in [FC]^-$ then by [5] there is a compact normal subgroup N such that $G/N \in [FIA]^-$. As mentioned in [8, p. 281], $\Pr C^*(G/N)$ is an open subset of $\Pr C^*(G)$. Since $\Pr C^*(G/N)$ contains P_e, it follows from the first part that P_e has a modular neighborhood which is Hausdorff.

$A(G)$ and $C^*(\lambda(A(G)))$

In order to prove our main theorems we will need some lemmas concerning Hilbert algebras. We begin by mentioning some general results; the reader can consult [12] for elementary facts. For our own reasons, however, we will not employ the "standard" notations used in [2], [12]. If A is a Hilbert algebra, we shall let λ (resp., ρ) denote the left (resp., right) regular representation of A on H, the Hilbert space completion of A:

$$\lambda(a)x = ax, \qquad \rho(a)x = xa^*, \qquad a, x \in A.$$

λ and ρ are then extended continuously to bounded operators on H. The letter U will be reserved in this context for unitary operators on the von Neumann algebra $U(A)$ generated by $\{\lambda(a): a \in A\}$. $\|\cdot\|$ will denote the operator uniform norm, while $\|\cdot\|_2$ will denote the Hilbert space norm. Finally, $C^*(\lambda(A))$ will mean the completion of $\{\lambda(a): a \in A\}$ with respect to the operator norm.

It is easy to see that $\|\lambda(a)\| = \|\rho(a)\|$, $a \in A$. In fact, a trivial calculation shows $(\lambda(a)x)^* = \rho(a)\, x^*$; since $x \mapsto x^*$ is norm preserving on H we have

$$\|\lambda(a)\| = \sup_{\|x\|_2 \leqslant 1} \|\lambda(a)x\|_2 = \sup_{\|x\|_2 \leqslant 1} \|(\lambda(a)x)^*\|_2 = \sup_{\|x\|_2 \leqslant 1} \|\rho(a)(x^*)\|_2 = \|\rho(a)\|.$$

Henceforth, when we are speaking of a Hilbert algebra A, it will be assumed to be full.

LEMMA 5. *Let $\{a_\nu\}$ be a net of elements in A such that $\{\lambda(a_\nu)\}$ is Cauchy in the operator norm and so that the numbers $\|a_\nu\|_2$ are uniformly bounded. Then $\{a_\nu a_\nu^*\}$ converges to an element of A.*

Proof. We must show $\{a_\nu a_\nu^*\}$ converges with respect to the norm $\|\cdot\|'$ defined on A by $\|a\|' = \|a\|_2 + \|\lambda(a)\|$, in which A is a Banach algebra (see [12, 1.15]). Convergence of $\{\lambda(a_\nu a_\nu^*)\}$ in the operator norm is clear. On the other hand, choose an index ν_0 so that $\nu, \mu \geqslant \nu_0$ imply $\| \lambda(a_\nu) - \lambda(a_\mu)\| < \epsilon$, where $\epsilon > 0$ is given. By assumption, $\| a_\nu \|_2 \leqslant M$ for all ν and for some positive number M. Then

$$\begin{aligned}\| a_\nu a_\nu^* - a_\mu a_\mu^* \|_2 &\leqslant \| a_\nu a_\nu^* - a_\nu a_\mu^* \|_2 + \| a_\nu a_\mu^* - a_\mu a_\mu^* \|_2\\ &\leqslant \|(\rho(a_\nu) - \rho(a_\mu))\, a_\nu \|_2 + \|(\lambda(a_\nu) - \lambda(a_\mu))\, a_\mu^* \|_2\\ &\leqslant \| \rho(a_\nu) - \rho(a_\mu)\| \, \| a_\nu \|_2 + \| \lambda(a_\nu) - \lambda(a_\mu)\| \, \| a_\mu^* \|_2,\\ &\leqslant \epsilon M + \epsilon M = 2\epsilon M.\end{aligned}$$

THEOREM 6. *Let A be a full Hilbert algebra and suppose that the center of $C^*(\lambda(A))$ is nonzero. Then the center of A is nonzero.*

Proof. Let $\{U_i\colon 1 \leqslant i \leqslant n\}$ be a finite set of unitary operators in $U(A)$ and let $\{t_i\colon 1 \leqslant i \leqslant n\}$ be real numbers, $0 \leqslant t_i \leqslant 1$, such that $\sum_{i=1}^{i=n} t_i = 1$. Since the elements of the form $\{\lambda(a)\colon a \in A\}$ constitute a two-sided ideal in $U(A)$ [12, 1.8], for a fixed $a \in A$ there is an $a' \in A$ such that (§) $\lambda(a') = \sum_{i=1}^{i=n} t_i U_i^* \lambda(a)\, U_i$. We show $\| a' \|_2 \leqslant \| a \|_2$. Let g and h denote elements of A; then

$$\| a' \|_2 = \sup_{\|\lambda(g)\| \leqslant 1 \;\|h\|_2 \leqslant 1} |(a, hg)|,$$

since the set of all products hg with $\| \lambda(g)\| \leqslant 1$ and $\| h \|_2 \leqslant 1$ is dense in the unit ball in H. This follows from the existence of a bounded approximate identity in A (see [12, 2.12]). We calculate

$$\begin{aligned}|(a', hg)| &= |(a', \rho(g^*)h \,| = |(a', \rho(g)^*h)|\\ &= |(\rho(g)\, a', h)| = |(\lambda(a')\, g^*, h)|\\ &\leqslant \sum_{i=1}^{i=n} t_i \,|(U_i^*\lambda(a)\, U_i g^*, h)|\\ &\leqslant \sum_{i=1}^{i=n} t_i \,|(\lambda(a)\, U_i g^*, U_i h)|\\ &\leqslant \sum_{i=1}^{i=n} t_i \,|(\rho((U_i g^*)^*)a, U_i h)|\\ &\leqslant \sum_{i=n}^{i=n} t_i \,\| \rho(U_i g^*)^* a \|_2 \, \| U_i h \|_2\\ &\leqslant \sum_{i=1}^{i=n} t_i \,\| \rho(U_i g^*)^* \| \, \| a \|_2 \, \| U_i h \|_2\\ &\leqslant \sum_{i=1}^{i=n} t_i \,\| a \|_2 = \| a \|_2,\end{aligned}$$

where we are using in the next to last line the fact that

$$\| \rho(U_i g^*)^* \| = \| \rho(U_i g^*)\| = \| \lambda(U_i g^*)\| = \| U_i \lambda(g^*)\|$$
$$\leqslant \| U_i \| \| \lambda(g)^* \| \leqslant \| \lambda(g)\| \leqslant 1.$$

Let K_a be the convex set generated by the elements $\{U^*\lambda(a)U: U \in U(A),\ U \text{ unitary}\}$; that is, K_a consists of all elements of the form (§). Denote by K_a^- the closure of K_a with respect to the operator norm. According to [2, Theorem 1, p. 253], $K_a^- \cap Z(U(A))$ is nonvoid, where $Z(U(A))$ is the center of the von Neumann algebra $U(A)$. Let $T \in K_a^- \cap Z(U(A))$ and let $\{a_\nu\}$ be a net of elements of A such that $\{\lambda(a_\nu)\} \subseteq K_a$ and $\{\lambda(a_\nu)\}$ converges to T in the operator norm. Our calculation above shows $\| a_\nu \|_2 \leqslant \| a \|_2$, so the hypotheses of Lemma 5 apply. Thus $\{a_\nu a_\nu^*\}$ converges to an element $b \in A$, and in fact $\lambda(b) = TT^*$, so b is in the center of A.

It remains only to show that $a \in A$ can be chosen so that $b \neq 0$; for this we must use the hypothesis that there is a nonzero central element B in $C^*(\lambda(A))$. We may suppose $\| B \| = 1$. Choose $0 < \epsilon < \frac{1}{4}$ and let $a \in A$ satisfy $\| B - \lambda(a)\| < \epsilon$. For $U \in U(A)$, U unitary, we observe

$$\| U^*\lambda(a)U - \lambda(a)\| \leqslant \| U^*\lambda(a)U - U^*BU \| + \| U^*BU - \lambda(a)\|$$
$$\leqslant \| U^* \| \| \lambda(a) - B \| \| U \| + \| B - \lambda(a)\| < 2\epsilon,$$

where we have used $U^*BU = U^*UB = B$. Now let $a' \in A$ with $\lambda(a')$ having the form (§) above. Then

$$\| \lambda(a') - \lambda(a)\| = \left\| \sum_{i=1}^{i=n} t_i U_i^*\lambda(a)\, U_i - \lambda(a) \right\|$$
$$\leqslant \sum_{i=1}^{i=n} t_i \| U_i^*\lambda(a)\, U_i - \lambda(a)\| \leqslant \sum_{i=1}^{i=n} t_i 2\epsilon = 2\epsilon.$$

It follows that for $T \subseteq K_a^- \cap Z(C^*(\lambda(A)))$, $\| \lambda(a) - T \| \leqslant 2\epsilon$, hence

$$\| B - T \| \leqslant \| B - \lambda(a)\| + \| \lambda(a) - T \| < 3\epsilon.$$

In particular, $T \neq 0$, so $TT^* \neq 0$, and finally $b \neq 0$.

LEMMA 7. *Let G be unimodular and $H \subseteq G$ an open subgroup. Then $C^*(H)$ (resp., $U(H)$) can be viewed as a subalgebra of $C^*(G)$ (resp., $U(G)$).*

Proof. Choose Haar measures on H and G so that the restriction to H of Haar measure on G is Haar measure on H. Then if we identify each function defined on H with its trivial extension to G (i.e., by defining it to be zero on $G \sim H$), we have the inclusion $L^p(H) \subseteq L^p(G)$, $1 \leqslant p \leqslant \infty$. Thus to each $f \in L^1(H)$ there are associated two C^*-norms: one arising from $L^1(H) \subseteq C^*(H)$ and the other from $L^1(H) \subseteq L^1(G) \subseteq C^*(G)$. We will show the two are equal. Let χ_H be the characteristic function of H; then $\chi_H \in P_1(H)$ and $f = f\chi_H$ (pointwise product). We calculate

$$\begin{aligned}\|f\|_{C^*(G)} &= \sup_{\varphi \in P_1(G)} \langle f^* * f, \varphi\rangle^{1/2} = \sup_{\varphi \in P_1(G)} \langle f^* * f, \varphi\chi_H\rangle^{1/2} \\ &= \sup_{\psi \in P_1(H)} \langle f^* * f, \psi\rangle^{1/2} = \|f\|_{C^*(H)} .\end{aligned}$$

Henceforth we will denote the common value by $\|f\|$. Thus $C^*(H)$, which is the completion of $L^1(H)$ with respect to this norm, may be viewed as a subalgebra of $C^*(G)$.

The second assertion requires some explanation since $U(H)$ and $U(G)$ are algebras of operators on different Hilbert spaces. However, since $L^2(H) \subseteq L^2(G)$ there is a projection $E \in \mathscr{L}(L^2(G))$ of $L^2(G)$ onto $L^2(H)$. Let $B(G, H)$ be the von Neumann subalgebra of $U(G)$ generated by $\{\lambda_G(f): f \in L^1(H)\}$. It is easy to see $E \in B(G, H)^c$, the commutant. In fact, let $\{x_\mu\}$ be a set of right coset representatives for G with respect to H, so $G = \bigcup_\mu Hx_\mu$ (disjoint union) and let $g \in L^2(G)$. There is at most a countable number of cosets Hx_{μ_i} for which $g_{\mu_i} = (g \mid Hx_{\mu_i})^\sim$ is nonzero. Write i for μ_i. Now $g = \sum_{i \geqslant 0} g_i$ and $g_i = ((g_i)_{x_i^{-1}})_{x_i}$. We have $(g_i)_{x_i^{-1}} \in L^2(H)$, and $f * h_x = (f * h)_x$ for arbitrary $h \in L^2(G)$, $x \in G$. Thus $f * g = \sum_{i \geqslant 0} (f * (g_i)_{x_i^{-1}})_{x_i}$. Now if $H = H_{x_{i_0}}$ then $\lambda(f)\, Eg = f * g_{i_0}$. For $i \neq i_0$ we have support$(f * (g_i)_{x_i^{-1}})_{x_i} \subseteq H_{x_i} \subseteq G \sim H$, so that

$$E\lambda(f)g = (f * (g_{i_0})_{x_{i_0}^{-1}})_{x_{i_0}} = f * g_{i_0} .$$

Similarly, if $H \neq Hx_i$ for all i, $\lambda(f)\, Eg = E\lambda(f)g = 0$.

Next we show that the central support of E in $B(G, H)^c$ is the identity; i.e., that

$$B(G, H)^c \cdot L^2(H) = \{Th: T \in B(G, H)^c, h \in L^2(H)\}$$

is dense in $L^2(G)$. However, $B(G, H) \subseteq U(G)$ implies $B(G, H)^c \subseteq U(G)^c = V(G)$, where $V(G)$ contains (in fact is generated by) the right translation operators $\rho_G(x)$, $x \in G$.

Let $g \in L^2(G)$ and $\epsilon > 0$. Retaining the notation of the preceding paragraph, there is an n for which

$$\left\| g - \sum_{i=0}^{i=n} g_i \right\|_2 < \epsilon.$$

Since $(g_i)_{x_i^{-1}} \in L^2(H)$ we have only to observe $g_i = \rho(x_i)(g_i)_{x_i^{-1}}$, hence

$$\left\| g - \sum_{i=1}^{i=n} \rho(x_i)(g_i)_{x_i^{-1}} \right\|_2 < \epsilon.$$

It follows from [2, p. 18] that $B(G, H) \to B(G, H)E$, $T \mapsto TE$ is an isomorphism of von Neumann algebras. Now for $f \in L^1(H)$, $\lambda_G(f)E = \lambda_H(f)$. Thus if $\{f_\nu\} \subseteq L^1(H)$ is a net, $\{\lambda_G(f_\nu)\}$ converges in the ultraweak topology iff $\{\lambda_H(f_\nu)\}$ converges in the ultraweak topology [2, Corollary 1, p. 54]. Since the ultraweak closure of $\{\lambda_H(f): f \in L^1(H)\}$ is $U(H)$, this is to say $B(G, H)E = U(H)$. Henceforth when we write $U(H) \subseteq U(G)$ we will mean that $U(H)$ is isomorphic to the von Neumann algebra $B(G, H) \subseteq U(G)$.

LEMMA 8. *Let H and G be unimodular and $H \subseteq G$ be open. Then $A(H) = A(G) \cap L^2(H)$.*

Proof. Let $a \in A(G) \cap L^2(H)$. Then $\lambda_G(a)$ is a bounded operator, hence so is $\lambda_G(a)E = \lambda_H(a)$, where $E: L^2(G) \to L^2(H)$ is the projection. Since $a \in L^2(H)$, it follows $a \in A(H)$, since by definition the full Hilbert algebra $A(H)$ contains all bounded elements. Let $a \in A(H)$ and define $\lambda_G(a)$ by $\lambda_G(a)g = a * g$, $g \in L^2(G)$. We must show $\lambda_G(a)$ is bounded. As in the proof of Lemma 7 write $g = \sum_{i \geqslant 0} g_i$, $g_i = g \mid Hx_i$. Because H is open and the cosets are disjoint, the square of the L^2-norm of a function on G is the sum of the squares of the L^2-norms on the various cosets; also, since support$(a) \subseteq H$, support$(a * g_i) \subseteq Hx_i$. Thus we have

$$\begin{aligned}
\| \lambda_G(a)g \|_2^2 &= \sum_{i \geqslant 0} \| \lambda_G(a) g_i \|_2^2 = \sum_{i \geqslant 0} \| a * (g_i) \|_2^2 \\
&= \sum_{i \geqslant 0} \| (a * (g_i)_{x_i^{-1}})_{x_i} \|_2^2 = \sum_{i \geqslant 0} \| a * (g_i)_{x_i^{-1}} \|_2^2 \\
&= \sum_{i \geqslant 0} \| \lambda_H(a)(g_i)_{x_i^{-1}} \|_2^2 \leqslant \| \lambda_H(a) \|^2 \sum_{i \geqslant 0} \| (g_i)_{x_i^{-1}} \|_2^2 \\
&\leqslant \| \lambda_H(a) \|^2 \sum_{i \geqslant 0} \| g_i \|_2^2 \leqslant \| \lambda_H(a) \|^2 \| g \|_2^2 .
\end{aligned}$$

Thus $\lambda_G(a)$ is a bounded operator on $L^2(G)$ by left multiplication by an element $a \in L^2(H) \subseteq L^2(G)$, hence $a \in A(G) \cap L^2(H)$.

Let $a \in A(H)$; by Lemma 7 there is a $T \in B(G, H)$ with $TE = \lambda_H(a)$. We claim $\lambda_G(a) = T$. Since by Lemma 8, $a \in A(G)$, hence $\lambda_G(a) \in U(G)$, $\lambda_G(a)$ commutes with the operators $\rho(x)$, $x \in G$. It is enough to show that $\lambda_G(a)$ and T agree on $\{g_x: g \in L^2(H), x \in G\}$, which is total in $L^2(G)$. But

$$\begin{aligned}\lambda_G(a)\, g_x &= \lambda_G(a)\, \rho(x) g = \rho(x)\, \lambda_G(a) g = \rho(x)\, \lambda_G(a)\, Eg \\ &= \rho(x)\, \lambda_H(a) g = \rho(x)\, Tg = T\rho(x) g = Tg_x\,.\end{aligned}$$

In connection with the next corollary we mention that $A(G)$ may be considered as the fulfillment of the Hilbert algebra $L^1 \cap L^2(G)$, and so we have $C^*(\lambda(L^1 \cap L^2(G))) \subseteq C^*(\lambda(A(G)))$, and the inclusion is in general strict. As an example consider the case $G = Z$, the integers. Here we have $C^*(\lambda(L^1 \cap L^2(Z))) = C^*(Z) \simeq C(T)$, the continuous functions on the circle group. On the other hand, since $L^1 \cap L^2(Z)$ has an identity δ_0, point mass at zero, the ideal $\lambda(A(G))$ is the entire von Neumann algebra $U(G)$. Thus we have $\lambda(A(G)) = C^*(\lambda(A(G))) = U(G) \simeq L^\infty(T, \mu)$, where μ is Haar mass on T. Furthermore, $L^\infty(T, \mu) \subseteq L^2(T, \mu) \simeq L^2(Z)$ by the Plancherel theorem.

COROLLARY 9. *Let H, G be unimodular and amenable and $H \subseteq G$ be open. Then $\lambda_G C^*(H) = \lambda_G C^*(G) \cap C^*(\lambda_G(A(H)))$.*

Proof. By the universal property of the C^*-algebras $C^*(H)$, $C^*(G)$, the representation λ_G of $L^1(H)$ and $L^1(G)$ extends to a representation of $C^*(H)$ and $C^*(G)$, respectively. Since G is amenable, λ_G is a faithful representation of $C^*(G)$, and hence also faithful on $C^*(H)$ (since by Lemma 7 the latter is a subalgebra).

That $\lambda_G C^*(H) \subseteq \lambda_G C^*(G) \cap C^*(\lambda_G(A(H)))$ is clear. Suppose $q \in C^*(G)$; we decompose $q = q_1 + q_2$ with $q_1 \in C^*(H)$ as follows. Let $\{f_\nu\} \subseteq L^1 \cap L^2(G)$ be a net converging to q in $C^*(G)$. Then if χ_H is the characteristic function of H, we claim $\{\chi_H f_\nu\}$ is Cauchy in $C^*(H)$.

$$\begin{aligned}\| \chi_H f_\nu - \chi_H f_\mu \| &= \sup_{\varphi \in P_1(H)} \langle (\chi_H f_\nu - \chi_H f_\mu)^* * (\chi_H f_\nu - \chi_H f_\mu), \varphi \rangle^{1/2} \\ &\leqslant \sup_{\varphi \in P_1(H)} \langle (f_\nu - f_\mu)^* * (f_\nu - f_\mu), \tilde{\varphi} \rangle^{1/2} \\ &\leqslant \| f_\nu - f_\mu \|.\end{aligned}$$

Let $q_1 = \lim_\nu \chi_H f_\nu$, $q_2 = \lim g_\nu$, where $g_\nu = (1 - \chi_H) f_\nu$.

Next we claim $C^*(\lambda_G(A(H))) \subseteq B(G, H)$. Clearly it is enough to show $\lambda_G(A(H)) \subseteq B(G, H)$. Let $a \in A(H)$; then $\lambda_G(a)E = \lambda_H(a) \in U(H) = B(G, H)E$, so the claim follows from the remark after Lemma 8.

Let $\lambda_G(q) \in \lambda_G(C^*(G)) \cap C^*(\lambda_G(A(H)))$ and write $q = q_1 + q_2$ as above. Then $\lambda_G(q)$, $\lambda_G(q_1) \in B(G, H)$, hence so is $\lambda_G(q_2)$. Thus

$$\| q_2 \| = \sup_{\substack{h_i \in L^2(H), \|h_i\|_2 \leqslant 1 \\ (i=1,2)}} (\lambda_H(q_2)\, h_1 , h_2).$$

Let $\epsilon > 0$ be given; there is an index ν_0 so that for $\nu \geqslant \nu_0$, $\| q_2 - g_\nu \| = \| \lambda_H(q_2 - g_\nu)\| < \epsilon$. If we now write

$$(\lambda_H(q_2)\, h_1 , h_2) = (\lambda_H(q_2 - g_\nu)\, h_1 , h_2) + (\lambda_H(g_\nu)\, h_1 , h_2),$$
$$h_i \in L^2(H), \qquad \| h_i \|_2 \leqslant 1 \qquad (i = 1, 2),$$

then the first term on the right-hand side is less than ϵ, and the other is zero, since support$(g_\nu) \subseteq G \sim H$. We conclude $q = q_1 \in C^*(H)$.

Henceforth we shall identify $C^*(G)$ and $C^*(\lambda(L^1(G))) = C^*(\lambda(L^1 \cap L^2(G)))$ in case G is amenable. Also, identifying $\lambda_G(C^*(G))$ with $C^*(G)$ as well as $\lambda_G(C^*(H))$ with $C^*(H)$ we can write Corollary 9 as simply $C^*(\lambda(A(H))) \cap C^*(G) = C^*(H)$.

In the following proof, when we refer to the "strong form" of Theorem 6 we will mean given $T \in Z(C^*(\lambda(A)))$ and $\epsilon > 0$ there is an $a \in Z(A)$ with $\| T - \lambda(a)\| < \epsilon$.

Proof of Theorem 1. Following the notation of [1, III], let $B = \Pr C^*(G)$, M the topological complete regularization of B, and $\varphi: B \twoheadrightarrow M$ the canonical surjection. By assumption the point P_e has a Hausdorff neighborhood U in B. Since B is locally compact, there are compact neighborhoods U_0, U_1 with $P_e \in U_0 \subseteq \operatorname{Int} U_1 \subseteq \operatorname{Int} U$. We claim $\varphi \mid U_0$ is a homeomorphism. Since U_0 is compact and φ is continuous and M is by definition Hausdorff, it remains only to show $\varphi \mid U_0$ is injective. Now given $P \neq Q \in U_0$, one can construct a continuous function f with $f(P) \neq f(Q)$ and support$(f) \subseteq U_1$. Since B and M have the same continuous functions, $\varphi(P)$ and $\varphi(Q)$ must be distinct in M.

Since $C^*(G)/k(U)$ has an identity and $U_0 \subseteq U$, it follows $C^*(G)/k(U_0)$ is likewise an algebra with identity. Let $a \in C^*(G)$ be any element which maps to the identity in $C^*(G)/k(U_0)$ under the canonical map $C^*(G) \to C^*(G)/k(U_0)$. In the notation of [1, 8.13], $\check{a}(\varphi(P)) = 1_P$ for all $P \in U_0$, where 1_P is the identity in $C^*(G)/P$, $P \in U_0$. Let $g \in C^b(M)$, $0 \leqslant g \leqslant 1$, $g(\varphi(P_e)) = 1$, and support$(g) \subseteq \varphi(U_0)$. The section $\check{b}$ defined by $\check{b}(m) = g(m)\,\check{a}(m)$, $m \in M$, is then either zero or a multiple of the identity in $C^*(G)/m$. But then $\check{b} \in Z(\Gamma_0(\pi''))$, the center of $\Gamma_0(\pi'')$. Since $C^*(G) \to \Gamma_0(\pi'')$ is an isometric star algebra isomorphism, $b \in Z(C^*(G))$.

Since G is amenable, $C^*(G) = C^*(\lambda(L^1 \cap L^2(G))) \subseteq C^*(\lambda(A(G))) \subseteq U(G)$. Now $\lambda(L^1 \cap L^2(G))$ is weakly dense in $U(G)$, so in particular $C^*(G)$ is weakly dense in $C^*(\lambda(A(G)))$. Thus $Z(C^*(G)) \subseteq Z(C^*(\lambda(A(G))))$, and since $Z(C^*(G)) \neq (0)$, the hypotheses of Theorem 6 are satisfied. So there is a nonzero $c \in Z(A(G))$, hence $\lambda(c) \in Z(U(G))$. Thus $\lambda(c) = \lambda(x)\,\lambda(c)\,\lambda(x^{-1}) = \lambda({}_{x^{-1}}c_x)$. It follows easily that $c = {}_{x^{-1}}c_x$ a.e. Thus the function $f(x) = |c(x)|^2$ is nonzero, central, and in $L^1(G)$. From [10] we have $G \in [IN]$.

Let $H \subseteq G$ be the open normal subgroup consisting of elements with precompact conjugacy classes. If $H \neq G$ then by the Gelfond–Raikov theorem there is a $\pi' \neq \pi_e'$, π_e' the trivial representation of G/H and $\pi' \in \widehat{(G/H)}$. Let π, π_e denote the liftings of these representations to G. If $\ker \pi = \ker \pi_e$ then $C^*(G)/\ker \pi$ is commutative, so π is one-dimensional. Thus π and π_e would be two multiplicative linear functionals with the same kernel, hence equal. Thus if $P = \ker \pi$, $P \neq P_e$, so from our assumption that P_e has a relatively Hausdorff neighborhood U there is a compact neighborhood V of P_e such that $P \notin V$.

Let $a \in C^*(G)$ be as above. We construct an element $b \in Z(C^*(G))$ by $\check{b} = g\check{a}$ as above, except now we require that support$(g) \subseteq W$, $W = \varphi(U_0 \cap V)$. If $b_1 = bb^*$, then support$(\|\check{b}_1\|) \subseteq W$ and $b_1(\varphi(P_e)) = 1_{P_e}$. By the strong form of Theorem 6, b_1 can be approximated by a net $\{\lambda(b_\nu)\}$, $\{b_\nu\} \subseteq Z(A(G))$. Since each b_ν is a central function in $L^2(G)$, support$(b_\nu) \subseteq H$. Thus $b_\nu \in A(G) \cap L^2(H) = A(H)$ (cf. Lemma 8). Hence $b_1 \in C^*(\lambda(A(H))) \cap C^*(G) = C^*(H)$ (cf. Corollary 9).

Let $f \in L^1(H)$ satisfy $\|\lambda(f) - b_1\| < \frac{1}{4}$. (If we consider f and b_1 as elements of $C^*(H)$, we may simply write $\|f - b_1\| < \frac{1}{4}$.) Now $\|\check{b}_1(\varphi(P_e))\| = 1$ and $\|\check{b}_1(\varphi(P))\| = 0$, hence $\|\check{f}(\varphi(P_e))\| > \frac{3}{4}$ and $\|\check{f}(\varphi(P))\| < \frac{1}{4}$. As P is not required to lie in U_0, it may happen that $\varphi(P) = \varphi(P')$ for some $P \neq P'$, $P' \in B \sim U_0$. In any case we at least have $\|\check{f}(\varphi(P))\| \geqslant \|\hat{f}(P)\| = \|\pi(f)\|$. Now

$$(\pi(f)h, k) = \int_G f(x)(\pi(x)h, k)\,dx = \int_H f(x)(\pi(x)h, k)\,dx = (h, k)\int_H f(x)\,dx.$$

Thus

$$\|\pi(f)\| = \sup_{\substack{h,k \in H_\pi \\ \|h\|,\|k\| \leqslant 1}} |(\pi(f)h, k)| = \left|\int_H f(x)\,dx\right|.$$

On the other hand, $\hat{f}(\varphi(P_e)) = \hat{f}(P_e) = \pi_e(f) = \int_H f(x)\,dx$. This implies $\|\hat{f}(\varphi(P)\| \geqslant \|\hat{f}(\varphi(P_e))\|$, which is absurd. We conclude that $G = H$, so $G \in [FC]^-$.

We mention in connection with the proof of the above theorem that the first conclusion, $G \in [IN]$, could have been attained under weaker hypotheses: Instead of assuming that Pr $C^*(G)$ is regular at P_e it would have been enough to assume that the topological complete regularization M of Pr $C^*(G)$ has a modular neighborhood. This would be sufficient to construct a central function in $C^*(G)$, and the reasoning would then proceed as above.

LEMMA 10. *Let A be a C^*-algebra. If* Pr A *is completely regular then A has an approximate identity consisting of central functions.*

Proof. We sketch two proofs. Let $P \in \Pr A$; as in the proof of Theorem 1 we can find an $a_P \in Z(A)$ with $\hat{a}_P(P) = 1_P$, $\| a \| = 1$. Let $U_P = \{P' \in \Pr A \colon \| \hat{a}_P(P')\| > \frac{1}{2}\| a \|\}$. Suppose now a compact subset $K \subseteq \Pr A$ is given. K can be covered by a finite number of U_P's; let $\{U_i \colon 1 \leqslant i \leqslant n\}$ cover K and $V = \bigcup_{i=n}^{i=n} U_i$. Then $\| \sum_{i=1}^{i=n} \hat{a}_i \hat{a}_i^*(P)\| \geqslant \delta > 0$ for some positive number δ, for all $P \in V$. Now let f be a continuous function on Pr A, $0 \leqslant f \leqslant 1$, $f(P) = 0$ for $P \in K$ and $f(P) = 1$ for $P \in \Pr A \sim V$. Hence $\| \sum_{i=1}^{i=n} \hat{a}_i \hat{a}_i^*(P)\| + f(P) \geqslant \delta'$ for all P Pr A, where $\delta' = \min\{1, \delta\}$. Since $P \mapsto \| \sum_{i=1}^{i=n} \hat{a}_i \hat{a}_i^*(P)\|$ is continuous on Pr A (because Pr A is Hausdorff), we can define the section u by

$$\hat{u}(P) = \left(\left\| \sum_{i=1}^{i=n} \hat{a}_i \hat{a}_i^*(P) \right\| + f(P)\right)^{-1} \left(\sum_{i=1}^{i=n} a_i a_i^*(P)\right).$$

Thus to each compact $K \subseteq \Pr A$ we can associate a central element u_K with $\| u_K \| = 1$ and $\hat{u}_K(P) = 1_P$, $P \in K$. The net $\{u_K \colon K \subseteq \Pr A, K \text{ compact}\}$, ordered by inclusion, is a central approximate identity for A.

The result can also be obtained as follows. Let A_1 be the C^*-algebra obtained in the usual way by the adjunction of an identity to A. By [11, 2.7.3], Pr A completely regular implies Pr A_1 completely regular, and in fact Pr A_1 is just the one-point compactification of Pr A. By [1, 8.14], $Z(A_1)$ is identified with the continuous functions on Pr(A_1). But $A = \{a \in A_1 \colon \hat{a}$ vanishes at infinity$\}$. In particular, since Pr A is locally compact Hausdorff, given a compact set $K \subseteq \Pr A$ there is a continuous function $\hat{u}_k$ on Pr A with $\hat{u}_k \mid K = 1$, $\| \hat{u}_K \| = 1$.

Proof of Theorem 2. By the universal property of the C^*-enveloping algebra $C^*(G)$, the left regular representation λ of $L^1(G)$

gives rise to a representation, which we also denote by λ: $C^*(G) \to C^*(\lambda(L^1(G))) = C^*(\lambda(L^1 \cap L^2(G))) \subseteq C^*(\lambda(A(G)))$. The central approximate identity in $C^*(G)$ which exists by virtue of Lemma 10 is mapped by λ to a central approximate identity in $C^*(\lambda(L^1 \cap L^2(G)))$. Given $a \in C^*(\lambda(L^1 \cap L^2(G)))$ and $\epsilon > 0$ there is a $u = u^* \in Z(C^*(\lambda(L^1 \cap L^2(G))))$, $\| u \| = 1$, with $\| au - u \| < \epsilon$. By Theorem 6 there is a $v = v^* \in Z(A(G))$ with $\| \lambda(v) - u \| < \epsilon/\| a \|$, and one can even require $\| \lambda(v) \| = 1$. Then

$$\| \lambda(v)a - a \| \leqslant \| \lambda(v)a - ua \| + \| ua - a \| < 2\epsilon.$$

Let $B(G) = \{v \in Z(A(G)): v = v^*, \| \lambda(v) \| = 1\}$. Let $T \in U(G)$ and suppose $T\lambda(v) = 0$ for all $v \in B(G)$. We then claim $T = 0$. If $T \neq 0$ there is an $h \in L^2(G)$ with $T^*h \neq 0$. Let $\{f_n\} \subseteq L^1 \cap L^2(G)$ be a sequence such that $\| \lambda(f_n)h - T^*h \|_2 = \epsilon_n \to 0$. We can do this since $\lambda(L^1 \cap L^2(G))$ is weakly dense, hence strongly dense in $U(G)$. Let $\{v_n\} \subseteq B(G)$ be such that $\| \lambda(f_n)\, \lambda(v_n) - \lambda(f_n) \| < 1/n$. It follows that

$$\begin{aligned} \| \lambda(f_n)\, \lambda(v_n)h - T^*h \|_2 &\leqslant \| \lambda(f_n)\, \lambda(v_n)h - \lambda(f_n)h \|_2 \\ &\quad + \| \lambda(f_n)h - T^*h \|_2 \\ &< (1/n) \| h \|_2 + \epsilon_n . \end{aligned}$$

Then $T\lambda(f_n)\, \lambda(v_n)h = T\lambda(v_n)\, \lambda(f_n)h \to TT^*h$, and since $T\lambda(v_n) = 0$, $TT^*h = 0$. On the other hand, $0 \neq \| T^*h \|_2^2 = (T^*h, T^*h) = (TT^*h, h)$, so $TT^*h \neq 0$. Thus given $0 \neq T \in U(G)$ there is a $v \in B(G)$ with $T\lambda(v) \neq 0$. But $T\lambda(v) = \lambda(Tv)$, and so $Tv \neq 0$. By [2, Proposition 3, p. 90] each $v \in B(G)$ is a trace element for $U(G)$; i.e., the function ω_v: $U(G)^+ \to R^+$, $T^*T \mapsto \omega_v(T^*T) = (T^*Tv, v)$ defines a finite normal trace. From the calculation above it follows that for every $0 \neq T^*T \in U(G)^+$ there is a finite normal trace ω_v with $\omega_v(T^*T) \neq 0$. Thus $U(G)$ is a finite von Neumann algebra, and according to [3, 13.10.5] this is equivalent to $G \in [SIN]$.

Concluding Remarks

Let G be σ-compact, unimodular, and amenable, and suppose $P_e \in \Pr C^*(G)$ has a relatively Hausdorff neighborhood. By Theorem 1, $G \in [FC]^-$, and so by [8], $\Pr C^*(G)$ is Hausdorff. Thus we see, at least to a certain extent, the topology of a neighborhood of $P_e \in \Pr C^*(G)$ determines the topology of $\Pr C^*(G)$. Note, however, the regularity of P_e does not imply every point $P \in \Pr C^*(G)$ is a regular point. Indeed, as we see from Corollary 3, if every point is regular then $G \in [FIA]^-$.

Let $G \in [FC]^-$ and $N \subseteq G$ a compact normal subgroup so that $G/N \in [FIA]^-$. As we have noted (see Remark 4), $\Pr C^*(G/N) \subseteq \Pr C^*(G)$ is open. Two questions arise: Are all regular points (resp., maximal modular ideals) in $\Pr C^*(G)$ contained in $\Pr C^*(G/N)$, and if not, then how are the regular points (resp., maximal modular ideals) in $\Pr C^*(G)$ distributed?

If in Theorem 1 one relaxes the requirement that P_e be a regular point and require merely that it have a relatively Hausdorff neighborhood in $\Pr C^*(G)$, then the question of what can be said about G remains open. We observe, however, that a C^*-algebra can have a Hausdorff primitive dual space and at the same time have a center consisting of (0). An example is the algebra of compact operators on an infinite-dimensional Hilbert space, whose one-point primitive dual space is trivially Hausdorff and whose center is (0). Here our method of attack would fail, and the investigation in this case would seem to lie somewhat deeper.

We mention finally the possibility of lifting the assumption of unimodularity in Theorems 1 and 2. One way this could conceiavbly be done would be to prove Lemma 5, Theorem 6, etc., for quasi-Hilbert algebras.

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